Shear Alfvén waves in turbulent plasmas

Manuel Núñez
Departamento de Análisis Matemático, Universidad de Valladolid, 47005 Valladolid, Spain
(Received 5 October 2001; published 7 March 2002)

The rate of decay of shear Alfvén waves along a magnetic field line of a diffusive plasma grows with the number of nodes of the initial perturbation. It is reasonable to think that the energy dissipation produced by this decay will be small if the perturbation was localized in a small set. This does not happen in turbulent plasmas: transport of the oscillation by the flow involves the whole domain. A general relation is obtained proving that the global energy dissipation is bounded below by an exponential of the number of nodes of any shear Alfvén wave along a segment of any field line of the average magnetic field.

DOI: 10.1103/PhysRevE.65.036417 PACS number(s): 52.35.Ra, 52.35.Bj, 94.30.Tz

I. INTRODUCTION

Alfvén waves may be viewed as elastic oscillations of magnetic field lines within a plasma. They were one of the first and still most important successes of the magnetohydrodynamical (MHD) description of plasmas, and their importance in many aspects of plasma physics, ranging from the stability of magnetically confined plasmas to magnetospheric problems, is unquestionable. Shear Alfvén waves are particularly intuitive in ideal (without kinetic or magnetic diffusivity) plasmas: they may be interpreted as transversal perturbations of a background magnetic field, and can be strongly localized near particular field lines. By this reason they were proposed for plasma heating purposes: see, e.g., [1]. The idea was that an electromagnetic wave of the right frequency would enter in resonance with a particular set of field lines away from the plasma boundary in order to inject energy there. While it is true that fusion and astrophysical plasmas have an extremely low diffusivity, this is nonetheless positive and Alfvén waves are modified by it: they eventually decay unless a forcing is present, and diffuse away from its original location. Nevertheless the concept has proved extremely robust and Alfvén waves have been identified almost everywhere, from the solar wind [2] to pseudodiffusion in liquid metals [3].

The frequency of shear Alfvén waves may, in principle, be arbitrarily high: it is connected to the number of nodes of the perturbed field along the background field line, rather in a manner similar to the frequency of the sound produced by a plucked string depends on the number of the up and down oscillations of the string. For perturbations of an incompressible static equilibrium with magnetic field $\mathbf{B}_0$, the dispersion relation of a perturbed field $\mathbf{b}=\mathbf{b}_0 \exp(i\mathbf{k}\cdot\mathbf{x}+i\omega t)$ is

$$\omega=-\frac{(\nu+\eta)k^2}{2}\pm i\left((\mathbf{B}_0\cdot\mathbf{k})^2-\frac{(\nu-\eta)^2k^4}{4}\right)^{1/2}, \qquad (1)$$

where, for simplicity, we have normalized the magnetic permeability and the plasma density to 1; ν stands for the viscosity and η for the resistivity of the plasma. If both coincide, it is apparent that we may get an arbitrarily high frequency (the imaginary part of ω) with a vector $\mathbf{k}$ large and parallel to $\mathbf{B}_0$, although the higher the frequency, the faster it decays.

In axisymmetric configurations the frequency of Alfvén waves grows with the azimuthal mode [4], essentially the same phenomenon. In the same spirit, field lines connecting null points are particularly unstable by accumulation of nodes at the ends [5,6].

Although as we have seen that diffusivity damps the energy of Alfvén waves, the more rapidly it does, the higher the number of nodes, it is obvious that if the wave did not have much energy in the first place (by being localized near a small set of field lines), energy dissipation will remain small. We may ask what could happen in turbulent plasmas, where the diffusion of Alfvén waves is enhanced by a chaotic fluid flow. It is clear that localization will not work there: the perturbation will extend to the whole turbulent region much more quickly than the time scale of simple resistive diffusion. In these conditions, how can one relate the number of nodes of a shear Alfvén wave along a single field line with the total energy dissipation? The question is not purely academic: plasmas with a large-scale magnetic field and small-scale turbulent velocity and field are very relevant (see, e.g., [7]). An extremely important phenomenon, the Alfvén effect, consists in that small-scale velocity $\mathbf{u}$ and field $\mathbf{b}$ tend to become identical or opposed ($\mathbf{u}=\pm\mathbf{b}$), precisely as in the shear Alfvén waves. This has been explained (although not with total rigor) by arguing that the average field $\mathbf{B}_0$ acts as a field guide for $\mathbf{u}$ and $\mathbf{b}$, who behave as Alfvén perturbations [8]. In particular this implies equipartition of energy: $u^2=b^2$. Unquestionably the Alfvén effect exists [9,10] and it plays an important role in several astrophysical phenomena. Moreover, it is the reason why it has been argued [11,12] that the Kolmogorov statistics for homogeneous hydrodynamic turbulence should be modified for plasmas: the decay at the inertial range is slower, $k^{-3/2}$ against $k^{-5/3}$. It seems worthwile to study how Alfvén waves dissipate energy in turbulent plasmas.

To address this question, we will first use some rigorous results on the Gevrey regularity of the solutions of the Navier-Stokes and MHD equations to find that these solutions may be extended analytically to a three-dimensional complex space. Some classical function theory will provide us with a bound of the number of zeroes of an analytic function in a ball in terms of its maximum size at the boundary. This may be applied to the number of nodes of a shear Alfvén wave along a field line. Thus we may estimate the number of nodes by the maximum size of the complex extension of the magnetic field. So far all the results are fully rigorous, but to find this maximum we need to accept the previously mentioned Iroshnikov-Kraichnan energy statistics for MHD turbulence. These will yield a bound of the maximum in terms of physically relevant parameters: resistivity, energy dissipation and Alfvén velocity. As a consequence we will obtain a lower bound on the global energy dissipation in terms of the number of nodes of a shear Alfvén wave along *any* background magnetic field line in the turbulent region. This result is possible thanks to the mixing properties of chaotic flows, which in more than one sense are simpler than more structured ones.

II. GENERAL ANALYSIS

Although the concept is more general, for our purposes it is enough to recall that a periodic function f with Fourier coefficients $\{f_{\mathbf{k}}:\mathbf{k}\in Z^3\}$ is said to satisfy a Gevrey condition if for some $\sigma>0$,

$$\sum_{\mathbf{k}\in Z^3} e^{2\sigma k}|f_{\mathbf{k}}|^2<\infty.$$

It is known that if the velocity $\mathbf{u}$ satisfies the Navier-Stokes equations in a periodic box $[0,2\pi]^3$, its Fourier expansion

$$\mathbf{u}(t,\mathbf{x})=\sum_{\mathbf{k}\in Z^3}\mathbf{u}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}} \tag{2}$$

satisfies a Gevrey condition for some $\sigma>0$,

$$\sum_{\mathbf{k}\in Z^3} e^{2\sigma k}|\mathbf{u}_{\mathbf{k}}(t)|^2<\infty, \tag{3}$$

where for any multi-index $\mathbf{k}$, $k=|k_1|+|k_2|+|k_3|$ [13]. This means that the function $\mathbf{x}\to\mathbf{u}(t,\mathbf{x})$ may be analytically extended to the product of three bands $([0,2\pi]\times(-\sigma,\sigma))^3$ in the complex space C^3. This result is always true even with an added forcing, provided it too satisfies a Gevrey condition, up to some time depending on the initial $\mathbf{u}(0)$, and also for as long as the solution and its gradient remain bounded in the L^2 norm. The proof of this theorem depends on some formal properties of the Navier-Stokes equations that are also valid for the MHD system (see, e.g., [14]). In fact the result has been later generalized to a wider class of equations that include the MHD ones [15]. Therefore, both velocity $\mathbf{u}$ and magnetic field $\mathbf{B}$ may be analytically extended.

Since we will be interested in the number of zeroes of some component B_i of $\mathbf{B}$, we recall one of the most important tools for these tasks: the Jensen formula, which we repeat for convenience (see, e.g., [16] for a proof). Let f be an analytic function in a disk $D(0,R)$, and let $\alpha_1,\ldots,\alpha_N$ be its zeroes in $\overline{D}(0,R)$, repeated as many times as their multiplicities. Then

$$|f(0)|\prod_{n=1}^{N}\frac{R}{|\alpha_n|}=\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\ln|f(Re^{i\theta}|d\theta)\right).$$

Denote by $n(R)$ the number of zeroes of f within the ball $D(0,R/2)$, and let $M(R)$ be the maximum of $|f|$ at the circle $S(0,R)$. Then it easily follows

$$n(R)\leqslant\frac{1}{\ln 2}\ln\frac{M(R)}{|f(0)|}. \tag{4}$$

Since we wish to bound the number of zeroes of B_i at some segment contained in the original box, we will need to imbed the segment into some ball within a linear subspace of C^3 of complex dimension 1, and also within the domain of analiticity of B_i. The real zeroes will certainly be no more than all the zeroes in the ball; the counterpart is that we need to know the maximum of $|B_i|$ not only at the real points, but also at the complex ones. Hence one cannot simply consider the size of $\mathbf{B}$ at the physical points, which is a measurable quantity. While it is true that the complex extension of $\mathbf{B}$ is determined by its real values, there is no useful estimate of its maximum modulus in terms of its restriction to the original box. At this point we must make use of some *a priori* form for the Fourier expansion. This is provided, for homogeneous turbulence in fluids, by the Kolmogorov distribution of energy. Let us remember (see, e.g., [17]) that the Fourier modes of the velocity are divided, in increasing order, in the injection range $k<k_{in}$, where the dynamics are governed by the forcing (assumed to be a large-scale one); the inertial range $k_{in}\leqslant k<k_d$, where energy is transferred to smaller scales via a direct cascade, follows; finally the dissipative range, $k_d\leqslant k<\infty$, where viscous dissipation is predominant. The size of the Fourier coefficients in each of these ranges is a different function of the energy dissipation $\varepsilon=-dE/dt$, where E is the kinetic energy

$$E=\frac{1}{2}\int_\Omega u^2dV.$$

The injection range is naturally dependent on the forcing and cannot be determined in general. Within the inertial range the identity

$$|\mathbf{u}_{\mathbf{k}}|^2=C_K\varepsilon^{2/3}k^{-5/3} \tag{5}$$

is satisfied. C_K, the Kolmogorov-Obukhov constant is determined by independent arguments. Its value varies somewhat with experiments: $C_K\simeq1.4-2$.

In the dissipative range, $\mathbf{u}_{\mathbf{k}}$ decreases exponentially

$$|\mathbf{u}_{\mathbf{k}}|^2=De^{-\alpha k}, \tag{6}$$

where D is chosen so as to make $|\mathbf{u}_{\mathbf{k}}|$ continuous in $\mathbf{k}$ at the boundary between ranges. The boundary k_d is given by the Kolmogorov microscale

$$k_d=(\varepsilon\nu^{-3})^{1/4}.$$

Notice that such a family of coefficients certainly are Gevrey summable, but we have now a much more precise expression for them. As asserted in the Introduction, turbulent plasmas follow a slightly different distribution. As before, the Fourier modes of both the velocity and the field are divided in three ranges: the injection one, where they depend directly on the forcing, and the inertial and dissipative ranges. In the inertial range the energy (kinetic plus magnetic) satisfies

$$|\mathbf{u_k}|^2+|\mathbf{B_k}|^2=Ck^{-3/2}\varepsilon^{1/2}v_A^{1/2}, \tag{7}$$

where v_A, the Alfvén velocity, is essentially the magnitude of a large-scale average magnetic field $\mathbf{B}_0$ assumed to exist, and C is a different constant. C is less universal than the Kolmogorov-Obukhov constant as it depends on $\mathbf{B}_0$. ε is again the energy dissipation rate. In this case

$$\varepsilon=-\frac{1}{2}\frac{d}{dt}\int_\Omega u^2+B^2 dV.$$

As a matter of fact we will need only $|\mathbf{B_k}|^2\leqslant C\varepsilon^{1/2}v_A^{1/2}$. At the dissipative range the decay is exponential

$$|\mathbf{u_k}|^2+|\mathbf{B_k}|^2=De^{-\gamma k}. \tag{8}$$

Numerical simulations show $\gamma\simeq 4.8k_d^{-1}$ [7]. D is chosen so as to make the spectrum continuous at the limit point, given in this case by

$$k_d=\left(\frac{\varepsilon}{\eta^2 v_A}\right)^{1/3}.$$

This means that

$$D=C\varepsilon^{1/2}v_A^{1/2}k_d^{-3/2}e^{\gamma k_d}. \tag{9}$$

Notice that this means that the Gevrey exponent σ of Eq. (3) must satisfy $\sigma<\gamma$, and that $\mathbf{B}$ is defined in $([0,2\pi]\times(-\gamma,\gamma))^3$.

III. NODES OF ALFVÉN WAVES

Let us, therefore, assume that we are dealing with an incompressible turbulent plasma where the Iroshnikov-Kraichnan statistics hold. Let $\mathbf{B}_0$ be a large-scale average magnetic field, with the turbulence provided by a small-scale, fluctuating field $\mathbf{b}$: thus $\mathbf{B}=\mathbf{B}_0+\mathbf{b}$. Let us consider a field line of $\mathbf{B}_0$, which given the characteristics of this field, we may take as a straight segment $S=[\mathbf{x}_0-R\mathbf{e},\mathbf{x}_0+R\mathbf{e}]$ at least for a length $2R\leqslant 2\sigma$; $\mathbf{e}$ is a unit directional vector. Assume that at least in the segment S, $\mathbf{b}$ is given by shear Alfvén perturbations along $\mathbf{B}_0$. Thus there $\mathbf{B}=B_0\mathbf{e}+b\mathbf{e}'$, $\mathbf{e}'$ orthogonal to $\mathbf{e}$. Take $\mathbf{x}_0$ such that $|\mathbf{b}(\mathbf{x}_0)|=r>0$. All this is done for a fixed time t; we may change the segment for another t, provided its length is bounded below by $2R$. To find a bound for $|\mathbf{B}|$, we will estimate all the terms of

$$|\mathbf{B}(t,\mathbf{z})|\leqslant\sum_{\mathbf{k}}|\mathbf{B_k}(t)||\mathbf{z^k}|$$

associated to the inertial and dissipative ranges. For the inertial range, which does not include the constant term $\mathbf{k}=\mathbf{0}$,

$$\begin{aligned}\sum_{1\leqslant k<k_d}|\mathbf{B_k}||\mathbf{z^k}|&\leqslant\sum_{1\leqslant k<k_d}C^{1/2}\varepsilon^{1/4}v_A^{1/4}k^{-3/4}e^{\sigma k}\\&\leqslant\sum_{1\leqslant k<k_d}C^{1/2}\varepsilon^{1/4}v_A^{1/4}e^{\sigma k}\\&=C^{1/2}\varepsilon^{1/4}v_A^{1/4}\left[\left(\frac{4k_d^2}{e^\sigma-1}-\frac{4k_d}{e^\sigma-1}-\frac{8(k_d-1)}{(e^\sigma-1)^2}\right.\right.\\&\quad\left.\left.+\frac{8}{(e^\sigma-1)^3}\right)e^{\sigma k_d}-\frac{8}{(e^\sigma-1)^3}\right].\end{aligned} \tag{10}$$

We will assume that as usual in turbulent plasmas, the energy dissipation is not negligible and the resistivity is small. Then k_d is large and the dominant term in the sum above is

$$\frac{4C^{1/2}\varepsilon^{1/4}v_A^{1/4}}{e^\sigma-1}k_d^2 e^{\sigma k_d}.$$

As for the dissipative range, we may bound it by

$$\begin{aligned}\sum_{k_d\leqslant k<\infty}|\mathbf{B_k}||\mathbf{z^k}|&\leqslant\sum_{k_d\leqslant k<\infty}De^{(\sigma-\gamma)k}\\&=De^{(\sigma-\gamma)k_d}\left(\frac{4k_d^2}{1-e^{\sigma-\gamma}}-\frac{4k_d}{1-e^{\sigma-\gamma}}\right.\\&\quad\left.-\frac{8k_d}{(1-e^{\sigma-\gamma})^2}+\frac{8}{(1-e^{\sigma-\gamma})^3}\right).\end{aligned} \tag{11}$$

Now the dominant term is

$$\frac{4D}{1-e^{\sigma-\gamma}}k_d^2 e^{(\sigma-\gamma)k_d}=\frac{4C\varepsilon^{1/2}v_A^{1/2}}{1-e^{\sigma-\gamma}}k_d^{1/2}e^{\sigma k_d}.$$

Let us fix σ, say at $\sigma=\gamma/2$. Then obviously the contribution from the dissipative terms is negligible in comparison with our bound on the inertial range, since we have now k_d to a power of 1/2 instead of 2.

We will simply assume that the contribution of the injection terms is bounded by a constant N independent of the large values k_d and η^{-1}, which is reasonable. Let us apply Eq. (4) to the function $f(z)=\mathbf{B}(\mathbf{x}_0+z\mathbf{e})\cdot\mathbf{e}'=b(\mathbf{x}_0+z\mathbf{e})$, $R=\gamma/2$,

$$M(R)\leqslant N+\frac{4C^{1/2}\varepsilon^{1/4}v_A^{1/4}}{e^{\gamma/2}-1}k_d^2 e^{(\gamma/2)k_d}, \tag{12}$$

and therefore

$$\ln\frac{M(R)}{|f(0)|}\leqslant\ln\left[\frac{1}{r}\left(N+\frac{4C^{1/2}\varepsilon^{1/4}v_A^{1/4}}{e^{\gamma/2}-1}k_d^2 e^{(\gamma/2)k_d}\right)\right]. \tag{13}$$

The behavior of this function is governed by the large term $(e^{\gamma/2}-1)^{-1}k_d^2\simeq(2/4.8)k_d^3$. We may write

$$\ln \frac{M(R)}{|f(0)|} \leqslant \ln(k_d^3) - \ln r = \ln(\varepsilon \eta^{-2} v_A^{-1}) - \ln r. \quad (14)$$

Let $n(R)$ be the number of nodes of the Alfvén wave **b** along the segment $S = [\mathbf{x}_0 - (R/2)\mathbf{e}, \mathbf{x}_0 + (R/2)\mathbf{e}]$, where $R < \gamma/2$. We cannot take R as precisely $\gamma/2$ because we do not know for how long the field line of $\mathbf{B}_0$ may be taken as a straight segment. Jensen's formula for the number of zeroes of f yields

$$n(R) \leqslant \frac{1}{\ln 2} \ln\left(\frac{1}{r} \varepsilon \eta^{-2} v_A^{-1} \right), \quad (15)$$

which is our main estimate. It may also be written as

$$\varepsilon \geqslant r 2^{n(R)} \eta^2 v_A, \quad (16)$$

which asserts that the energy dissipation is larger than an amount depending on the analyticity radius (the exponent of decay of dissipative modes), times 2 to the number of nodes of any shear Alfvén wave along any field line of the average magnetic field; the resistivity to the power 2, and the Alfvén velocity. We emphasize that $n(R)$ is a purely local quantity connected to a single field line, whereas ε concerns the global energy. Such a close relation would be impossible in a plasma with a neatly ordered geometry, where Alfvén waves may remain localized.

IV. CONCLUSIONS

While it is obvious that Alfvén waves along a field line in a diffusive plasma decay more rapidly with the number of nodes of the perturbation, it seems reasonable that if the initial perturbation was localized along a single line, or a small set of them, the global dissipation of energy caused by the wave decay should be small. This is not correct in turbulent plasmas, as the chaotic flow disperses the wave much more efficiently than the magnetic diffusivity. In fact, by using some results of the analytic function theory and the Iroshnikov-Kraichnan energy spectrum distribution, an estimate may be found, which is below the total energy dissipation in terms of the number of nodes of a shear Alfvén wave along a segment of a single field line of the average magnetic field.

[1] L. Chen and A. Hasegawa, Phys. Fluids **17**, 1399 (1974).
[2] L.F. Burlage and J.M. Turner, J. Geophys. Res. **81**, 73 (1976).
[3] P.A. Davidson, *An Introduction to Magnetohydrodynamics*, Cambridge Texts in Applied Mathematics (Cambridge University Press, Cambridge, 2001).
[4] E. Hameiri, Commun. Pure Appl. Math. **38**, 43 (1985).
[5] M. Núñez, Phys. Plasmas **3**, 3288 (1996).
[6] M. Núñez, J. Plasma Phys. **3**, 537 (1998).
[7] D. Biskamp, *Nonlinear Magnetohydrodynamics* (Cambridge University Press, Cambridge, 1993).
[8] P.H. Diamond and G.G. Craddock, Comments Plasma Phys. Control. Fusion **13**, 287 (1990).
[9] M. Dobrowolny, A. Mangeney, and P. Veltri, Phys. Rev. Lett. **45**, 144 (1980).
[10] S. Sridhar, Ann. N.Y. Acad. Sci. **773**, 44 (1995).
[11] P.S. Iroshnikov, Sov. Astron. **7**, 566 (1964).
[12] R.H. Kraichnan, Phys. Fluids **8**, 1385 (1965).
[13] C. Foias and R. Temam, J. Funct. Anal. **87**, 359 (1989).
[14] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics* (Springer, New York, 1988).
[15] A. Ferrari and E.S. Titi, Commun. Partial Differ. Equ. **23**, 1 (1998).
[16] W. Rudin, *Real and Complex Analysis* (McGraw-Hill, New York, 1987).
[17] U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, 1995).